Mechanics of Solids

Introduction to the linearized theory of elasticity

- Boundary value problems: Definitions
- Saint- Venant’s Principle
- Superposition principle
- Methods of solutions
    - 2D elasticity problems
    - 3D elasticity problems

From the book: Mechanics of Continuous Media: an Introduction

1. J Botsis and M Deville, PPUR 2018
2. J Botsis, Class notes given during the course

Mechanics of Solids: Theory of Elasticity

Formulation of the boundary value problem

We consider a solid, of an isotropic homogeneous linearly elastic material, and subjected to body forces over it and prescribed displacements or tractions on its boundary. The following equations are available:

1. The 3 eqs of equilibrium:

$$\boldsymbol{\sigma}_{ij,j} + f_i = 0 \quad , \quad div\boldsymbol{\sigma} + \boldsymbol{f} = 0$$

($\boldsymbol{f}$ is body force vector)

2. The 6 equations defining the *strain-displacement relation:*

$$\boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad , \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$$

3. The 6 equations defining the *isotropic homogeneous stress-strain relation:*

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \quad , \quad \boldsymbol{\sigma} = \lambda tr\boldsymbol{\varepsilon}\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$$

There are 15 equations with 15 unkowns:

Three displacement components: u_i

Six strain components: ε_{ij}

Six stress components: σ_{ij}

→ The problem is well posed

We know that a linear elastic solid satisfies the second principle of thermodynamics and that there exists a potential function which, has a quadratic form in the strains (or the stresses).

$$W(\varepsilon_{ij}) = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij} \quad \text{with} \quad \sigma_{ij} = \frac{\partial W(\boldsymbol{\varepsilon}_{ij})}{\partial\boldsymbol{\varepsilon}_{ij}}$$

Navier's Equations

There are two ways to combine the 15 equations:

The first one is to consider the displacement components u_i as the unknowns.

Introduce $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ in

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

to obtain:

$$\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

Intoduce it to the equilibrium equations:

$$\sigma_{ij,j} + f_i = 0$$

to obtain:

$$(\lambda + \mu) u_{k,ki} + \mu u_{i,jj} + f_i = 0$$

These are the three Navier's Equations with the three displacement components u_i as the unknowns.

With the displacements known we go back to:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

to calculate the strains ε_{ij}

With the strains known we obtain the stresses σ_{ij}

from $\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i})$

Note that there is no need to satisfy the compatibility equations:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0$$

because we calculate ε_{ij} from u_i

Beltrami-Michell Compatibility Equations

The second way to combine the 15 equation is :

to consider the stress components σ_{ij} as unknowns.

Then we introduce the strain-stress relations:

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$$

In the compatibility equations

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0$$

to obtain:

$$(1+\nu)\sigma_{ij,kk} - \nu\sigma_{mm,nn}\delta_{ij} + \sigma_{pp,ij}$$
$$-(1+\nu)(\sigma_{iq,qj} + \sigma_{jr,ri}) = 0$$

From the equilibrium equations (take the derivatives):

$$\sigma_{iq,qi} + \sigma_{jr,ri} = -f_{i,j} - f_{j,i}$$

$$(1+\nu)\sigma_{ij,kk} - \nu\sigma_{mm,nn}\delta_{ij} + \sigma_{pp,ij}$$
$$+(1+\nu)(f_{i,j} + f_{j,i}) = 0$$

Taking the trace of the last equation we get:

$$(1-\nu)\sigma_{mm,nn} = -(1+\nu)f_{k,k}$$

Using it in the last equation we obtain $(\nu \neq 1)$ the <u>Beltrami-Michell compatibility Eqs:</u>

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{mm,ij} + f_{i,j} + f_{j,i} + \frac{\nu}{1-\nu}f_{n,n}\delta_{ij} = 0$$

In several problems the body forces can be assumed negligible. We have the simplification:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{mm,ij} = 0$$

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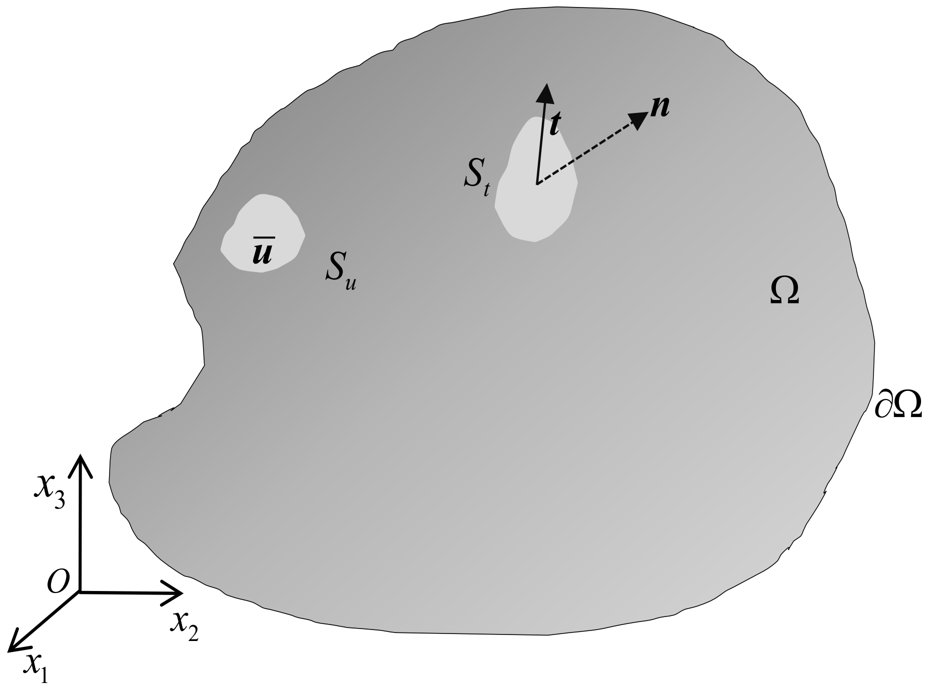

type I, or **mixed BVP**: we have to specify tractions and displacements on the corresponding parts of boundaries.

type II: we have to specify displacement on the Corresponding boundary conditions.

type III: we have to specify tractions on the corresponding part of boundaries:

BOUNDARY CONDITIONS

To solve the system of equations we need the appropriate boundary conditions: In general we have three of them.

We consider a body occupying a domain Ω in $\mathbb{R}^3$ with boundary $\partial\Omega$.

We divide the surface boundary into two parts so that:

$$\partial\Omega = S_u \cup S_t \quad , \quad S_u \cap S_t = \varnothing$$

S_u represents the part where displacements are prescribed:

$$u_i = \overline{u}_i \quad \text{on} \quad S_u$$

S_t represents the part where stress vector is prescribed:

$$t_i = \boldsymbol{\sigma}_{ij} n_j = \bar{t}_i \quad \text{on} \quad S_t$$

TYPE I or **mixed BVP**: we have to specify tractions and displacements on the corresponding parts of boundaries:

Navier Equations to solve $(\lambda + \mu)u_{k,ki} + \mu u_{i,jj} + f_i = 0$ over Ω

Subjected to Boundary Conditions:

Tractions: $t_i = \sigma_{ij} n_j = \bar{t}_i$ on S_t

$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$ ⟹ $\lambda u_{k,k} n_i + \mu(u_{i,j} + u_{j,i}) n_j = \bar{t}_i$ on S_t

Displacements: $u_i = \bar{u}_i$ on S_u

TYPE II: Displacement Boundary Conditions

we only have displacement boundary conditions $S_u \neq \varnothing \quad , \quad S_t = \varnothing$

Navier Equations to solve $(\lambda + \mu)u_{k,ki} + \mu u_{i,jj} + f_i = 0$ over Ω

Subjected to Boundary Conditions

Displacements: $u_i = \overline{u}_i$ on S_u

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TYPE III: we have to specify tractions on the corresponding part of boundaries:

Navier Equations to solve $(\lambda + \mu)u_{k,ki} + \mu u_{i,jj} + f_i = 0$ over Ω

Subjected to Boundary Conditions

Tractions: $t_i = \sigma_{ij} n_j = \overline{t_i}$ on S_t

$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$ ⟹ $\lambda u_{k,k} n_i + \mu(u_{i,j} + u_{j,i})n_j = \overline{t_i}$ on S_t

The traction BVP in terms of stress components : $S_u = \varnothing \quad , \quad S_t \neq \varnothing$

Here the following equations constitute the problem:

1. Equations equilibrium

$$\sigma_{ij,j} + f_i = 0 \quad \text{over} \quad \Omega$$

2. Stress compatibility equations:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{mm,ij} + f_{i,j} + f_{j,i} + \frac{\nu}{1-\nu} f_{n,n}\delta_{ij} = 0 \quad \text{over} \quad \Omega$$

3. Prescribed tractions on the surface:

$$t_i = \sigma_{ij} n_j = \overline{t_i} \quad \text{on} \quad S_t$$

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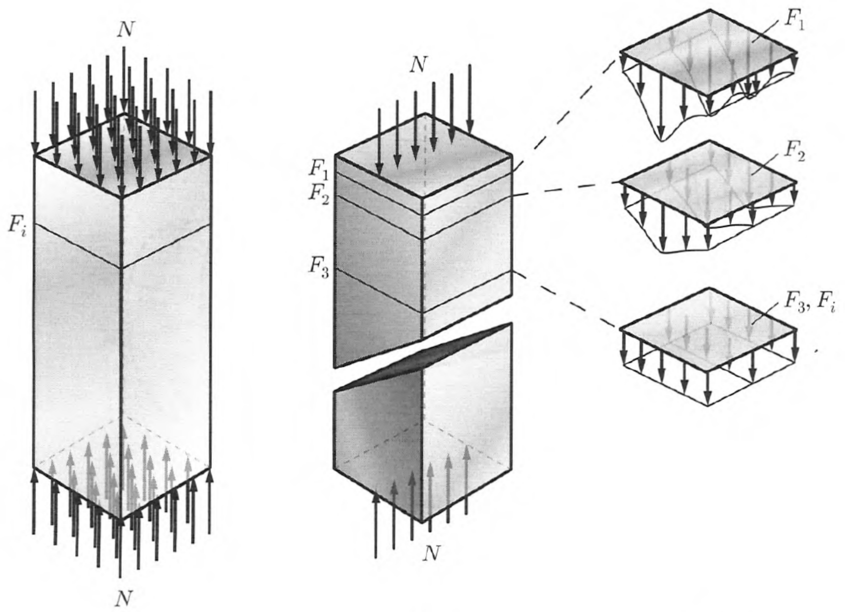
SAINT- VENANT'S PRINCIPLE

When forces are considered on the surface, the elasticity boundary problem is replaced with another for the same body but with the substitution of boundary conditions by statically equivalent conditions:

Example



SUPERPOSITION PRINCIPLE

In linear elastic theory, the 15 equations we have as well as the boundary conditions, are linear. This leads to the formulation of the superposition principle.

We consider a body occupying a domain Ω of $\mathbb{R}^3$ with boundary $\partial\Omega$.

I: it is subjected to surface forces $\overline{t_i}^{(1)}$ and body forces $\overline{f_i}^{(1)}$ and produce the stress field $\sigma_{ij}^{(1)}$.

II: It is subjected to surface forces $\overline{t_i}^{(2)}$ and body forces $\overline{f_i}^{(2)}$ to produce the stress field $\sigma_{ij}^{(2)}$.

The simultaneous application of

$(\overline{f_i}^{(1)} + \overline{f_i}^{(2)})$ and $(\overline{t_i}^{(1)} + \overline{t_i}^{(2)})$

will produce the stress field → $(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)})$

Mechanics of Solids: Theory of Elasticity

Plane Isotropic Linear Elasticity

Several important practical problems do not require the solution of the 3D problem for the state of stress and strain.

Because of the particular geometry of the solid and the form of the loads, the elasticity equations can be considered as functions of only two spatial variables.

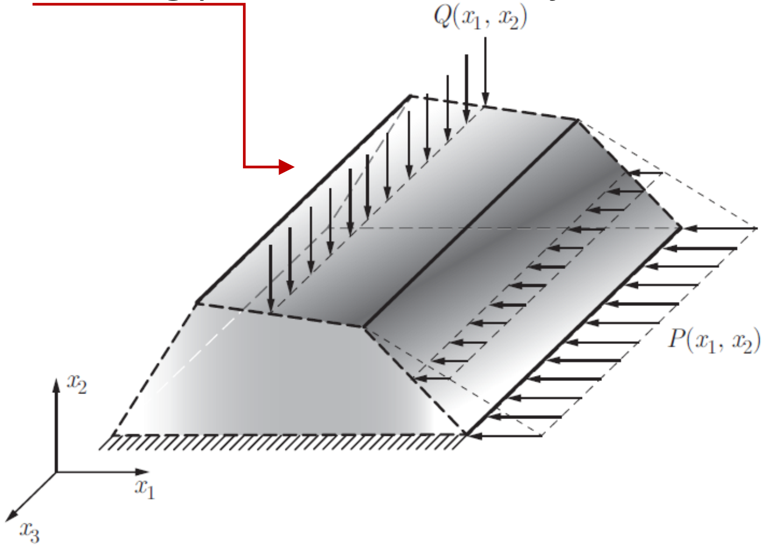
The problem is reduced to a plane problem.

Such plane problems are two:

1. Plane strain states
2. Plane stress states

State of Plane Strain

For long prismatic bars subjected to lateral forces:

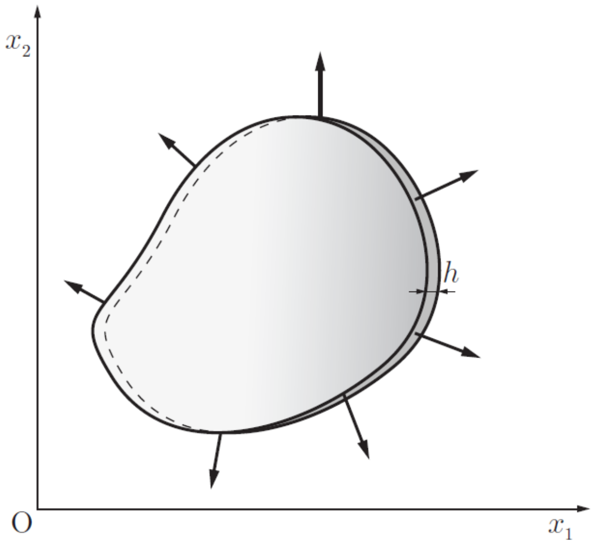


$$u_1 = u_1(x_1, x_2)$$
$$u_2 = u_2(x_1, x_2)$$
$$u_3 = u_3(x_3)$$

State of Plane Stress

For thin plates loaded in plane:

$$\sigma_{11} = \sigma_{11}(x_1, x_2)$$
$$\sigma_{22} = \sigma_{22}(x_1, x_2)$$
$$\sigma_{12} = \sigma_{12}(x_1, x_2)$$
$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$



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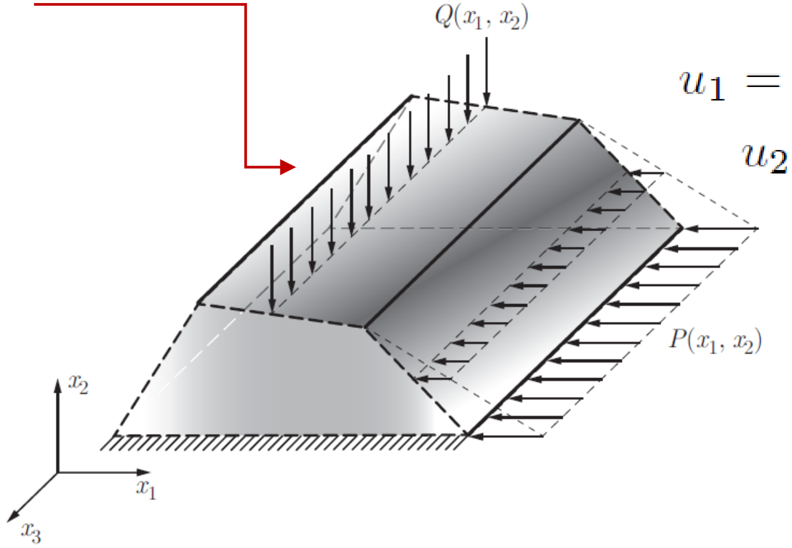
STATE OF PLANE STRAIN

For long prismatic bars subjected to lateral forces:



$$u_1 = u_1(x_1, x_2)$$
$$u_2 = u_2(x_1, x_2)$$
$$u_3 = u_3(x_3)$$

For a prismatic structure of infinite length or when its ends are fixed, we can assume in addition $u_3 = 0$ in each section.

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \qquad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = 0$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = 0$$

From Hook's Law

$$\boldsymbol{\sigma} = \frac{\nu E}{(1+\nu)(1-2\nu)} \operatorname{tr} \boldsymbol{\varepsilon} \boldsymbol{I} + \frac{E}{1+\nu} \boldsymbol{\varepsilon}$$

the non-zero stresses are $\sigma_{11}, \sigma_{22}, \sigma_{33}$ and σ_{12} (functions of x_1 and x_2):

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left(\varepsilon_{11}(1-\nu) + \nu\varepsilon_{22}\right)$$

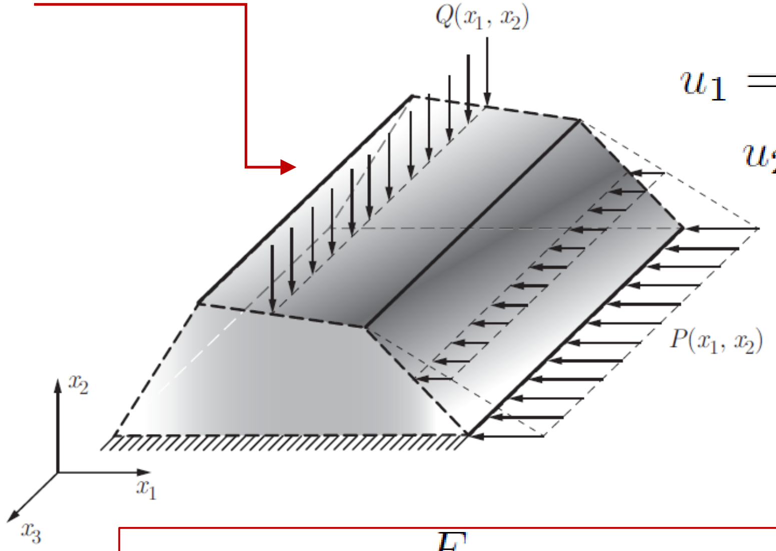
$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left(\varepsilon_{22}(1-\nu) + \nu\varepsilon_{11}\right)$$

$$\sigma_{12} = \frac{E}{(1+\nu)} \varepsilon_{12}$$

$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)} \nu(\varepsilon_{11} + \varepsilon_{22}) \, .$$

STATE OF PLANE STRAIN

For long prismatic bars subjected to lateral forces:



$$u_1 = u_1(x_1, x_2)$$
$$u_2 = u_2(x_1, x_2)$$
$$u_3 = u_3(x_3)$$

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left(\varepsilon_{11}(1-\nu) + \nu\varepsilon_{22}\right)$$
$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left(\varepsilon_{22}(1-\nu) + \nu\varepsilon_{11}\right)$$
$$\sigma_{12} = \frac{E}{(1+\nu)}\varepsilon_{12}$$
$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}\nu(\varepsilon_{11} + \varepsilon_{22})\,.$$

$$\varepsilon_{11} = \frac{1+\nu}{E}\left((1-\nu)\sigma_{11} - \nu\sigma_{22}\right)$$
$$\varepsilon_{22} = \frac{1+\nu}{E}\left((1-\nu)\sigma_{22} - \nu\sigma_{11}\right)$$
$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}\,.$$

$$\boldsymbol{\sigma}_{ij,j} + f_i = 0$$

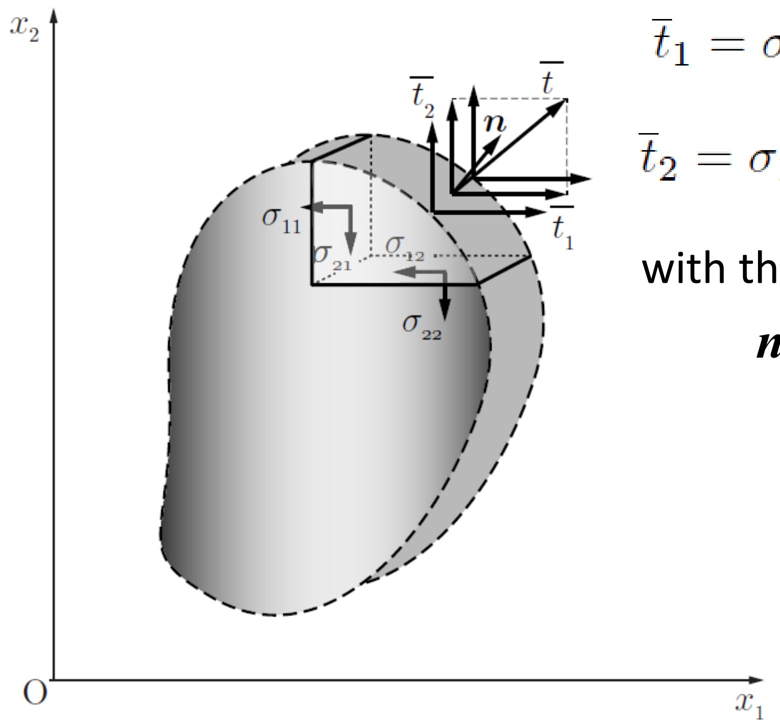
<u>Equations of equilibrium in plane strain:</u>

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + f_1 = 0$$
$$\frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + f_2 = 0$$

BOUNDARY CONDITIONS FOR PLANE ELASTICITY

The same assumptions apply here:

The surface forces $\bar{t}_1$ and $\bar{t}_2$ must be functions of only x_1 and x_2, with $\bar{t}_3 = 0$, in order that the strain be truly plane. For conditions of type II, we have:



$$\bar{t}_1 = \sigma_{11} n_1 + \sigma_{12} n_2$$

$$\bar{t}_2 = \sigma_{12} n_1 + \sigma_{22} n_2$$

with the normal vector

$$\boldsymbol{n}(n_1, n_2)$$

With the stresses chosen as unknowns, the compatibility equations must be satisfied. In the plane strain case, the only compatibility equation that is not automatically satisfied is:

$$\frac{\partial^2 \varepsilon_{11}}{\partial {x_2}^2} + \frac{\partial^2 \varepsilon_{22}}{\partial {x_1}^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

In Summary:

We have 8 equations and the following 8 unknowns:

$$\varepsilon_{11},\ \varepsilon_{22},\ \varepsilon_{12},\ \sigma_{11},\ \sigma_{22},\ \sigma_{12},\ u_1, u_2$$

should be calculated so that they satisfy and boundary conditions.

STATE OF PLANE STRAIN

The 8 equations can be reduced to 3 as follows.
The three stress-strain equations:

$$\varepsilon_{11} = \frac{1+\nu}{E}\left((1-\nu)\sigma_{11} - \nu\sigma_{22}\right)$$
$$\varepsilon_{22} = \frac{1+\nu}{E}\left((1-\nu)\sigma_{22} - \nu\sigma_{11}\right)$$
$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}\,.$$

are introduced in the compatibility equation,

$$\frac{\partial^2 \varepsilon_{11}}{\partial {x_2}^2} + \frac{\partial^2 \varepsilon_{22}}{\partial {x_1}^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

to obtain:

$$\frac{\partial^2}{\partial x_2^2}\left((1-\nu)\sigma_{11} - \nu\sigma_{22}\right) + \frac{\partial^2}{\partial x_1^2}\left((1-\nu)\sigma_{22} - \nu\sigma_{11}\right) = 2\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2}$$

Differentiate the equilibrium eqs (with x_1 and x_2) and add the result

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0$$

to obtain:

$$-2\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = \left(\frac{\partial^2 \sigma_{11}}{\partial {x_1}^2} + \frac{\partial^2 \sigma_{22}}{\partial {x_2}^2}\right) + \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)$$

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(\sigma_{11} + \sigma_{22}) = -\frac{1}{1-\nu}\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)$$

compatibility equation in terms of stresses

3 equations with 3 unknowns $\sigma_{11}\,, \sigma_{22}\,, \sigma_{12}$

Stress Function for Plane Strain Problems

The problem can be further reduced to one equation.

We assume that the volume forces are derived from a potential V:

$$f_i = -\frac{\partial V}{\partial x_i}, \quad i = 1, 2$$

It's easy to show that the following stresses,

$$\sigma_{11} = V + \frac{\partial^2 \Phi}{\partial {x_2}^2} \quad \sigma_{22} = V + \frac{\partial^2 \Phi}{\partial {x_1}^2}$$

$$\sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}$$

satisfy the two equilibrium eqs:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0 \qquad \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0$$

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(\sigma_{11} + \sigma_{22}) = -\frac{1}{1-\nu}\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)$$

$$\frac{\partial^4 \Phi}{\partial x_1^4} + 2\frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} + \frac{1-2\nu}{1-\nu}\left(\frac{\partial^2 V}{\partial {x_1}^2} + \frac{\partial^2 V}{\partial {x_2}^2}\right) = 0$$

$$\triangle\triangle\Phi + \frac{1-2\nu}{1-\nu}\triangle V = 0$$

For negligible body forces we have the biharmonic eq,

$$\triangle\triangle\Phi = 0$$

and

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial {x_2}^2}, \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial {x_1}^2}, \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}$$

Φ is the so-called **Airy stress function.**

Stress Function for Plane Strain Problems

Using a stress function the problem is solved as follows:

1: *identify the **Airy stress function** Φ, and verify the biharmonic equation:*

$$\triangle\triangle\Phi = 0$$

2: *calculate the stress components using:*

$$\sigma_{11} = \frac{\partial^2\Phi}{\partial {x_2}^2}\,,\quad \sigma_{22} = \frac{\partial^2\Phi}{\partial {x_1}^2}\,,$$

$$\sigma_{12} = -\frac{\partial^2\Phi}{\partial x_1 \partial x_2}$$

3: *calculate the strains from,*

$$\varepsilon_{11} = \frac{1+\nu}{E}\left((1-\nu)\sigma_{11} - \nu\sigma_{22}\right)$$

$$\varepsilon_{22} = \frac{1+\nu}{E}\left((1-\nu)\sigma_{22} - \nu\sigma_{11}\right)$$

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}\,.$$

4: *calculate the displacements from,*

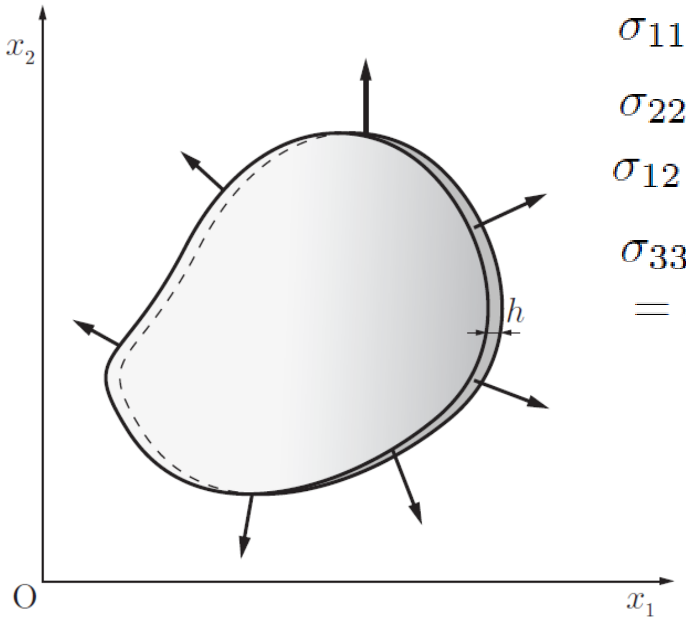
$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

Mechanics of Solids: Theory of Elasticity

STATE OF PLANE STRESS

For thin plates (i.e., along x_3) loaded in plane we assume that:



$$\sigma_{11} = \sigma_{11}(x_1, x_2)$$

$$\sigma_{22} = \sigma_{22}(x_1, x_2)$$

$$\sigma_{12} = \sigma_{12}(x_1, x_2)$$

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

<u>Equations of equilibrium in plane stress</u>

$$\boldsymbol{\sigma}_{ij,j} + f_i = 0$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0$$

From Hook's Law

$$\boldsymbol{\varepsilon} = -\frac{\nu}{E}\,\mathrm{tr}\,\boldsymbol{\sigma}\boldsymbol{I} + \frac{(1+\nu)}{E}\boldsymbol{\sigma}$$

and the non-zero stress components,

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}) \qquad \varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11}) \qquad \varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$$

$$\varepsilon_{13} = \varepsilon_{23} = 0$$

Invert

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22})$$

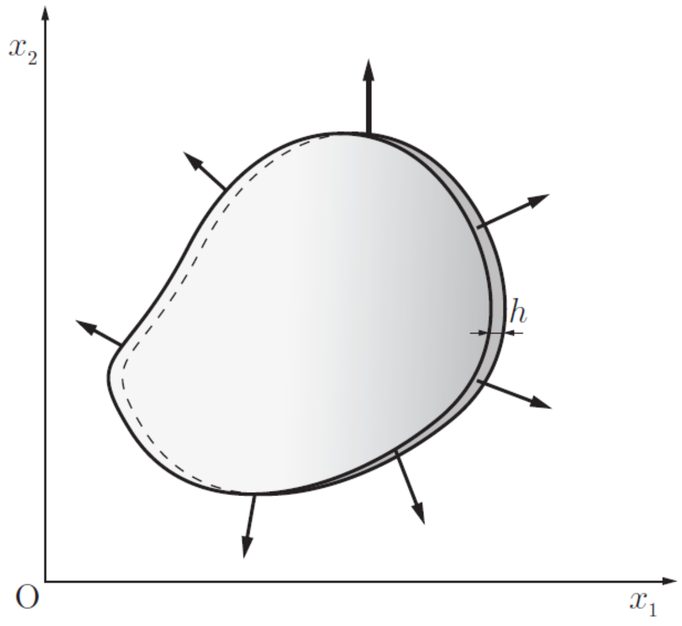
$$\sigma_{22} = \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11})$$

$$\sigma_{12} = \frac{E}{1+\nu}\varepsilon_{12}\,.$$

Mechanics of Solids: Theory of Elasticity

State of Plane Stress

For thin plates (i.e., along x_3) loaded in plane we assume that:



$$\sigma_{11} = \sigma_{11}(x_1, x_2) \quad \sigma_{22} = \sigma_{22}(x_1, x_2)$$

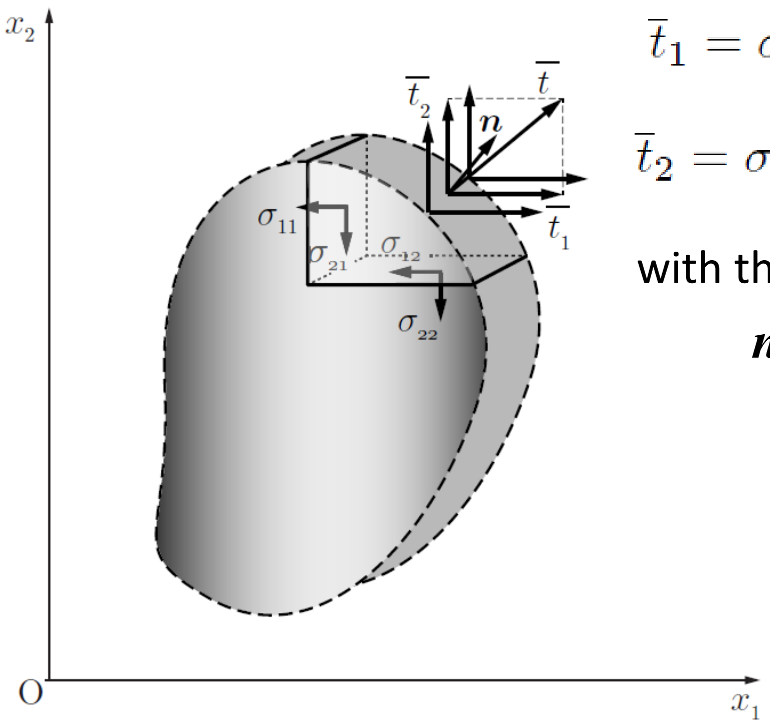
$$\sigma_{12} = \sigma_{12}(x_1, x_2)$$

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

Boundary Conditions for Plane Stress

The same assumptions apply here (as in Plane Strain).

The surface forces $\bar{t}_1$ and $\bar{t}_2$ must be functions of only x_1 and x_2, with $\bar{t}_3 = 0$, in order that the strain be truly plane. For conditions of type II, we have:



$$\bar{t}_1 = \sigma_{11} n_1 + \sigma_{12} n_2$$

$$\bar{t}_2 = \sigma_{12} n_1 + \sigma_{22} n_2$$

with the normal vector

$$\boldsymbol{n}(n_1, n_2)$$

State of Plane Stress

Compatibility Equation (the same as in plane strain):

$$\frac{\partial^2 \varepsilon_{11}}{\partial {x_2}^2} + \frac{\partial^2 \varepsilon_{22}}{\partial {x_1}^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} .$$

And the condition because ε_{33} in not zero:

$$\frac{\partial \varepsilon_{33}}{\partial x_1} = \frac{\partial \varepsilon_{33}}{\partial x_2} = \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = 0$$

Integration of the last relation imposes:

$$\varepsilon_{33} = A_0 + A_1 x_1 + A_2 x_2$$

In Summary:
As in the case of plane strain
we have 8 equations and the following 8 unknowns:

$$\varepsilon_{11},\ \varepsilon_{22},\ \varepsilon_{12},\ \sigma_{11},\ \sigma_{22},\ \sigma_{12} \quad , u_1 , u_2$$

should be calculated.

As in the case of plane strain, we can use the stresses as unknowns to reduce the equations from 8 to three. The substitution of strain components:

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}) \quad \varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11}) \quad \varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}$$

in the compatibility equation and the use of equilibrium results in:

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(\sigma_{11} + \sigma_{22}) = -(1+\nu)\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)$$

which with the 2 equilibrium eqs form the three eqs with three unknown stresses.

If we neglect the body forces and consider a stress function (Airy stress function) such as in the plane strain case,

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial {x_2}^2} , \ \sigma_{22} = \frac{\partial^2 \Phi}{\partial {x_1}^2} , \ \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}$$

we obtain the biharmonic equation $\triangle\triangle\Phi = 0$.

THERMAL STRESSES

We have for the strains from energetic considerations:

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} + \left(2\mu\alpha(T - T_0) - \frac{\lambda}{3\lambda + 2\mu}\,\mathrm{tr}\,\boldsymbol{\sigma}\right)\boldsymbol{I}\right)$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}\,\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\,\delta_{ij}\,.$$

In plane stress we have with $(\Delta T = T - T_o)$

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - v\sigma_{22}) + \alpha\Delta T;$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - v\sigma_{22}) + \alpha\Delta T;$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12}$$

→

$$\sigma_{11} = \frac{E}{1-v^2}(\varepsilon_{11} + v\varepsilon_{22}) - \frac{E\alpha\Delta T}{1-v};$$

$$\sigma_{22} = \frac{E}{1-v^2}(\varepsilon_{22} + v\varepsilon_{11}) - \frac{E\alpha\Delta T}{1-v};$$

$$\sigma_{12} = 2G\varepsilon_{12}$$

THERMAL STRESSES

Introduce the strains

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - v\sigma_{22}\right) + \alpha\Delta T;$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - v\sigma_{11}\right) + \alpha\Delta T;$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12}$$

in the compatibility equation

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2{}^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1{}^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}.$$

And combine it with the equilibrium equations with no body forces

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0 \quad \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0$$

Compatibility in terms of stresses

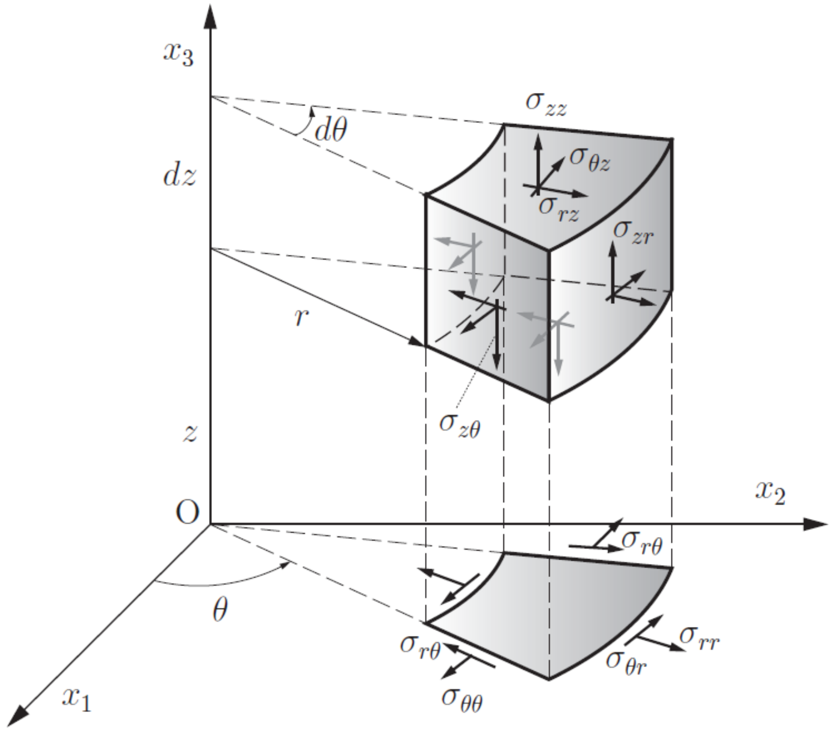
$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\sigma_{11} + \sigma_{22} + \alpha E \Delta T\right) = 0$$

Use the definitions

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2{}^2}, \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1{}^2}, \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}$$

$$\nabla^4 \Phi + \alpha E \nabla^2 (\Delta T) = 0$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates



Strain Components in Terms of Displacements

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\varepsilon_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right).$$

Biharmonic Equation

$$\nabla^4\Phi = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \times \left(\frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial \theta^2}\right) = 0$$

Equations of Equilibrium

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho b_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \rho b_\theta = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho b_z = 0\,.$$

Stress components

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial \theta^2} \qquad \sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2}$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial \theta}$$

Mechanics of Solids: Theory of Elasticity

IMPORTANT
In plane strain or plane stress we arrive at the same biharmonic equation:

$$\Delta\Delta\Phi = 0$$

Thus the stresses are the same in both cases. The difference between the two states appears in the stress-strain relations.

Direct analytical solutions of elasticity problems are not easy, and often, are not even possible.

Methods based on the rigorous use of applied mathematics are proposed to handle the some classes of problems.

Other techniques permit approximate solutions based on experience.

List of the methods most often used in linear elasticity.

Inverse Method. Here, the displacement or stress field is assigned to the body and we determine all the other quantities.
Method of Potential. *To simplify the elasticity equations, we introduce potential functions that yield the solution to Navier's equations or those for stress.*
Semi-Inverse Method. Here part of the stress and displacement fields are specified. Then, knowing these elements and applying Elasticity theory, the equations which must be satisfied by the remaining stresses and displacements are determined. Saint-Venant applied this method to bending and torsion of prismatic bars.
Complex Variable Methods. This method uses analytic functions defined in the complex plane to solve elasticity problems.
Variational Methods. Here the elasticity equations can be obtained by minimizing a generalized energy function.
Others. Other methods include integral transform methods and numerical approaches such as the finite element method.

Potential or Displacement Functions

To solve Navier's equations, potential or displacement functions are introduced in such a way that the displacement vector in Navier's equations is obtained from the derivatives of these functions.

- These potential functions are governed by Laplace's equation or the biharmonic equation, well known in mathematical physics.
- To advance further in that sense, we need the Helmholtz' decomposition.

Theorem:

Helmholtz' Theorem

A finite and continuous vector field $\boldsymbol{a}$, that is zero at inifity, can be represented as the sum of an irrotational field $\boldsymbol{b}$ and a solenoidal field $\boldsymbol{c}$:

$$\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{c} \qquad \text{with} \quad \nabla \times \boldsymbol{b} = 0 \quad \text{and} \quad div\ \boldsymbol{c} = 0$$

From vector calculus

1. There exists a scalar potential φ such that $\boldsymbol{b} = \boldsymbol{\nabla}\varphi$
2. There exists a potential vector function $\boldsymbol{\Psi}$ such that $\boldsymbol{c} = \boldsymbol{\nabla} \times \boldsymbol{\Psi}$

$$\boldsymbol{u} = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla} \times \boldsymbol{\Psi}$$

Potential or Displacement Functions

$$\boldsymbol{u} = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla} \times \boldsymbol{\Psi}$$

With this displacement we obtain:

$$\operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{\nabla}\varphi + \operatorname{div}(\boldsymbol{\nabla} \times \boldsymbol{\Psi}) = \operatorname{div} \boldsymbol{\nabla}\varphi + 0 = \nabla^2\varphi$$

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{u} &= \boldsymbol{\nabla} \times \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\Psi}) \\ &= \mathbf{0} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\Psi}) \\ &= \nabla(\operatorname{div} \boldsymbol{\Psi}) - \nabla^2\boldsymbol{\Psi} = -\nabla^2\boldsymbol{\Psi} \end{aligned}$$

(assume $\operatorname{div} \boldsymbol{\Psi} = 0$)

With these relations, the Navier Equations,

$$(\lambda + \mu)\nabla(\operatorname{div} \boldsymbol{u}) + \mu\triangle\boldsymbol{u} + \boldsymbol{f} = 0$$

become:

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\nabla^2\varphi) + \mu\boldsymbol{\nabla} \times (\nabla^2\boldsymbol{\Psi}) = 0$$

Every pair of functions satisfying this function produces a displacement given by:

$$\boldsymbol{u} = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla} \times \boldsymbol{\Psi}$$

that is a solution to Navier's equations. (The inverse statement also applies)

We know from kinematics that the infinitesimal strain tensor gives:

$$\operatorname{div} \boldsymbol{u} = \varepsilon_{ii} \longrightarrow \nabla^2\varphi = \varepsilon_{ii}$$

$$\frac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{u} = \omega_{32}\boldsymbol{e}_1 + \omega_{13}\boldsymbol{e}_2 + \omega_{21}\boldsymbol{e}_3$$

Mechanics of Solids: Theory of Elasticity

Potential or Displacement Functions

Lamé Strain Potential

Particular solutions of,

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\nabla^2\varphi) + \mu\boldsymbol{\nabla} \times (\nabla^2\boldsymbol{\Psi}) = 0$$

are produced from the two equations:

$$\nabla^2\varphi = \text{cnst} \quad ; \quad \nabla^2\boldsymbol{\Psi} = \text{cnst}$$

When $\nabla^2\varphi = \text{cnst}$ and $\boldsymbol{\Psi} = 0$

The displacement is given by: $\boldsymbol{u} = \boldsymbol{\nabla}\varphi$

or for simplicity $\boldsymbol{u} = \dfrac{1}{2\mu}\boldsymbol{\nabla}\varphi$

φ is called Lamé strain potential.

In terms of the Lamé potential

$$u_i = \frac{1}{2\mu}\varphi_{,i}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2\mu}\varphi_{,ij}$$

$$\varepsilon_{kk} = u_{k,k} = \frac{1}{2\mu}\varphi_{,kk}$$

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} = \frac{\lambda}{2\mu}\varphi_{,kk}\delta_{ij} + \varphi_{,ij}$$

For particular solutions in practice $\nabla^2\varphi = 0$ (i.e., Laplace equation and φ is harmonic) with:

$$\varphi(r,\theta) = Cr^n \cos n\theta, \quad r^2 = x_1^2 + x_2^2$$

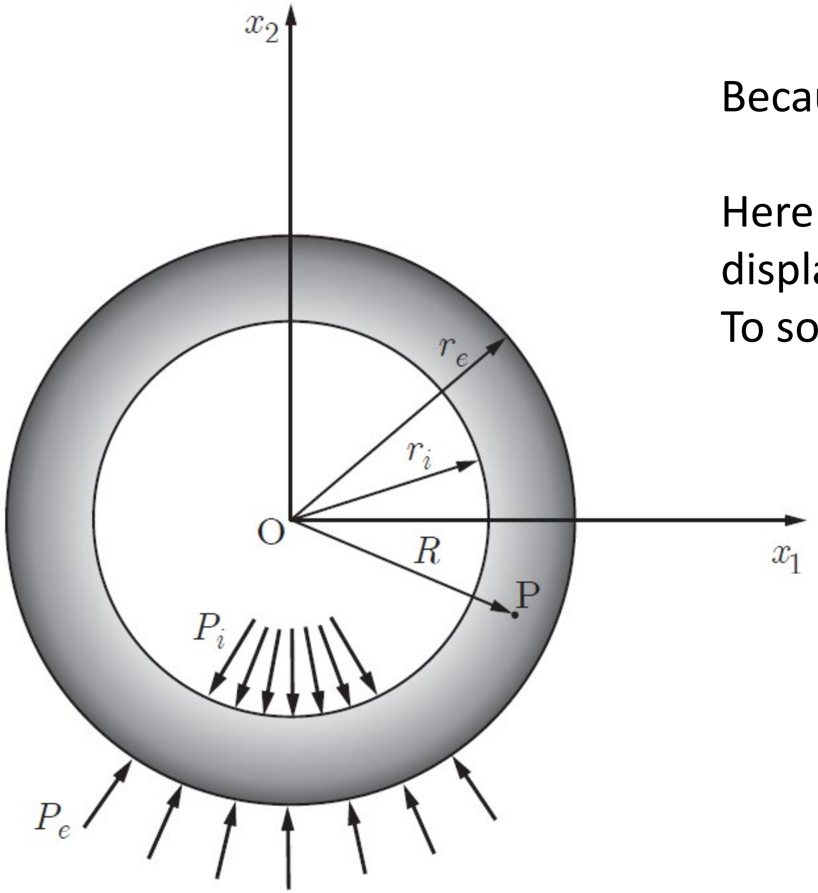
$$\varphi(r) = C\ln\frac{r}{K}, \quad r^2 = x_1^2 + x_2^2,$$

$$\varphi(\theta) = C\theta, \quad \theta = \tan^{-1}\frac{x_2}{x_1},$$

$$\varphi(R) = \frac{C}{R}, \quad R^2 = x_1^2 + x_2^2 + x_3^2 \ .$$

Potential or Displacement Functions

Example: *Hollow Sphere Subject to Internal and External Pressure*
(THICK-WALLED SPHERE)



Because of spherical symmetry, we use spherical coordinates (R, θ, φ).

Here all the shear stresses and strains are zero, and the only non-zero displacement is u_R .
To solve this problem, we use the potential function approach:

We combine two potentials to obtain the following:

$$\varphi(R) = \frac{C}{R} + DR^2$$

for the solution of the problem.

POTENTIAL OR DISPLACEMENT FUNCTIONS

Example: *Hollow Sphere Subject to Internal and External Pressure*
(THICK-WALLED SPHERE)

The chosen function

$$\varphi(R) = \frac{C}{R} + DR^2$$

satisfies Laplace Equation

$$\nabla^2 \varphi = \text{cnst}$$

Expressing all parameters in spherical coordinates (R, θ, φ) we obtain,

$$u_R = \frac{1}{2\mu}\left(-\frac{C}{R^2} + 2DR\right)$$

$$u_\theta = u_\varphi = 0$$

$$\varepsilon_{RR} = \frac{1}{2\mu}\left(\frac{2C}{R^3} + 2D\right), \quad \varepsilon_{\varphi\varphi} = \varepsilon_{\theta\theta} = \frac{1}{2\mu}\left(-\frac{C}{R^3} + 2D\right)$$

$$\varepsilon_{\theta\varphi} = \varepsilon_{\theta R} = \varepsilon_{\varphi R} = 0 \ .$$

Inserting these relations into Hook's law, we get

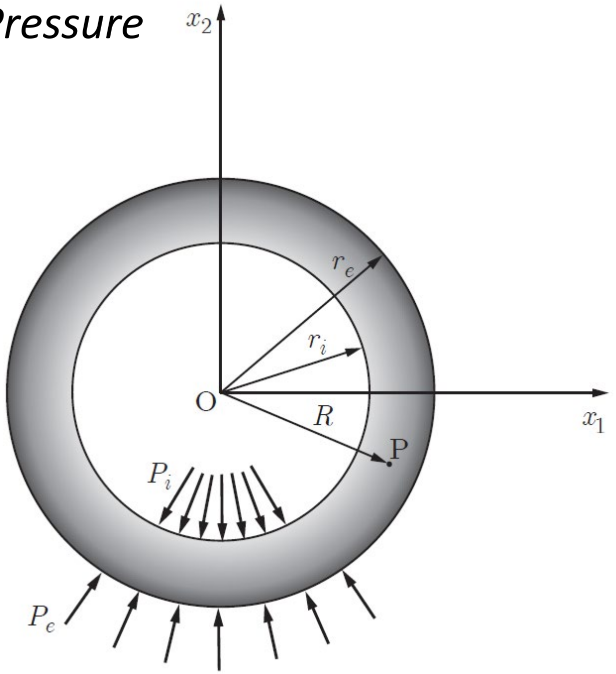
$$\sigma_{\varphi\varphi} = \sigma_{\theta\theta} = -\frac{C}{R^3} + 2\frac{1+\nu}{1-2\nu}D$$

$$\sigma_{RR} = \frac{2C}{R^3} + 2\frac{1+\nu}{1-2\nu}D,$$

$$\sigma_{\theta\varphi} = \sigma_{\theta R} = \sigma_{\varphi R} = 0 \ .$$

POTENTIAL OR DISPLACEMENT FUNCTIONS

Example: *Hollow Sphere Subject to Internal and External Pressure* (THICK-WALLED SPHERE)



The integration constants in the stresses

$$\sigma_{\varphi\varphi} = \sigma_{\theta\theta} = -\frac{C}{R^3} + 2\frac{1+\nu}{1-2\nu}D$$

$$\sigma_{RR} = \frac{2C}{R^3} + 2\frac{1+\nu}{1-2\nu}D$$

Are obtained with the help of BCs

$$\sigma_{RR} = -P_i \quad \text{at} \quad R = r_i$$
$$\sigma_{RR} = -P_e \quad \text{at} \quad R = r_e$$

$$C = \frac{1}{2}\frac{r_e^3 r_i^3 (P_e - P_i)}{r_e^3 - r_i^3}$$

$$D = \frac{1}{2}\frac{1-2\nu}{1+\nu}\frac{r_i^3 P_i - r_e^3 P_e}{r_e^3 - r_i^3}$$

Mechanics of Solids: Theory of Elasticity

Example: *Hollow Sphere Subject to Internal and External Pressure*(THICK-WALLED SPHERE)

$$\sigma_{RR} = \frac{1}{R^3}\frac{r_e^3 r_i^3 (P_e - P_i)}{r_e^3 - r_i^3} + \frac{r_i^3 P_i - r_e^3 P_e}{r_e^3 - r_i^3}$$

$$= -\frac{P_i\left(\frac{r_e^3}{R^3} - 1\right)}{\left(\frac{r_e^3}{r_i^3} - 1\right)} - \frac{P_e\left(1 - \frac{r_i^3}{R^3}\right)}{\left(1 - \frac{r_i^3}{r_e^3}\right)}$$

$$\sigma_{\varphi\varphi} = \sigma_{\theta\theta} = -\frac{1}{2R^3}\frac{r_e^3 r_i^3 (P_e - P_i)}{r_e^3 - r_i^3} + \frac{r_i^3 P_i - r_e^3 P_e}{r_e^3 - r_i^3}$$

$$= \frac{1}{2}\left(\frac{P_i\left(\frac{r_e^3}{R^3} + 2\right)}{\left(\frac{r_e^3}{r_i^3} - 1\right)} - \frac{P_e\left(\frac{r_i^3}{R^3} + 2\right)}{\left(1 - \frac{r_i^3}{r_e^3}\right)}\right).$$

$$u_R = \frac{R}{2\mu}\left(-\frac{1}{2R^3}\frac{r_e^3 r_i^3 (P_e - P_i)}{r_e^3 - r_i^3} + \frac{1-2\nu}{1+\nu}\frac{r_i^3 P_i - r_e^3 P_e}{r_e^3 - r_i^3}\right)$$

$$= \frac{R}{2\mu}\left(P_i \frac{\frac{1}{2}\frac{r_e^3}{R^3} + \frac{1-2\nu}{1+\nu}}{\frac{r_e^3}{r_i^3} - 1} - P_e \frac{\frac{1}{2}\frac{r_i^3}{R^3} + \frac{1-2\nu}{1+\nu}}{1 - \frac{r_i^3}{r_e^3}}\right).$$

When $r_e >> r_i$

$$\sigma_{RR} = -P_i \frac{r_i^3}{R^3} - P_e\left(1 - \frac{r_i^3}{R^3}\right)$$

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \frac{P_i}{2}\frac{r_i^3}{R^3} - \frac{P_e}{2}\left(\frac{r_i^3}{R^3} + 2\right)$$

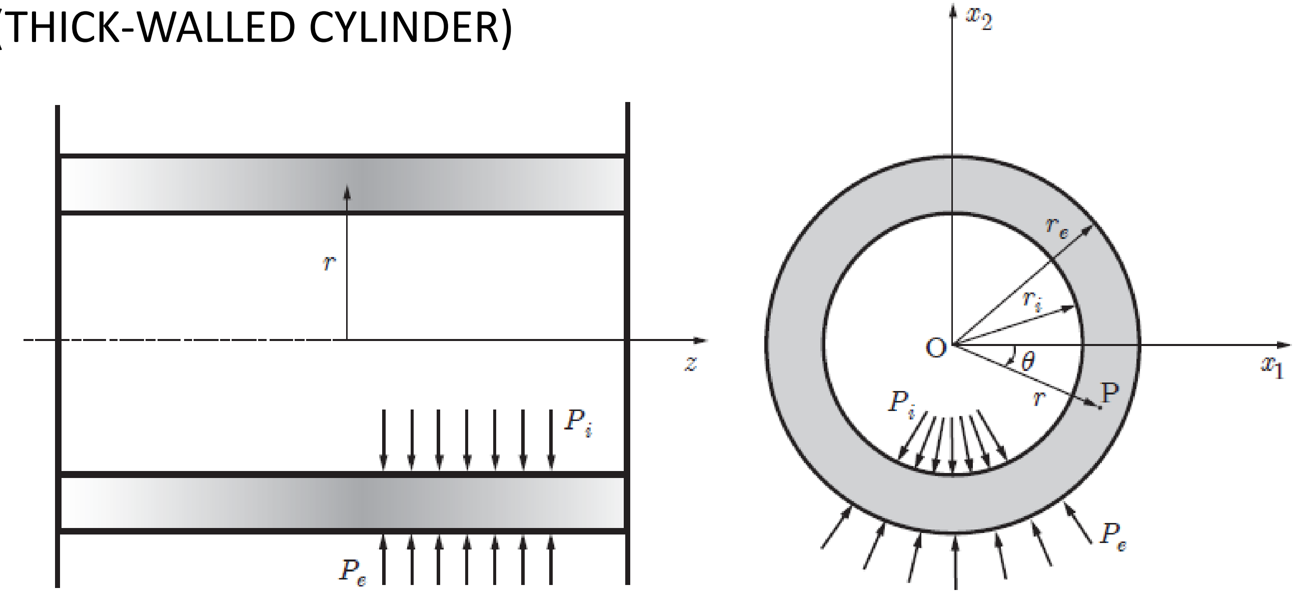
$$u_R = \frac{R}{2\mu}\left(\frac{P_i}{2}\frac{r_i^3}{R^3} - P_e\left(\frac{1-2\nu}{1+\nu} + \frac{1}{2}\frac{r_i^3}{R^3}\right)\right)$$

When $r_e - r_i$ is very small and P_e=0

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \frac{P_i r_i}{2(r_e - r_i)}$$

POTENTIAL OR DISPLACEMENT FUNCTIONS

Example: *Hollow Cylinder with Internal and External Pressures and fixed Ends Subject to Internal and External Pressure* (THICK-WALLED CYLINDER)



Boundary Conditions

$$\sigma_{rr} = -P_i, \quad \sigma_{r\theta} = 0 \quad \text{at} \quad r = r_i$$
$$\sigma_{rr} = -P_e, \quad \sigma_{r\theta} = 0 \quad \text{at} \quad r = r_e \ .$$

Because of cylindrical symmetry, we use cylindrical coordinates (r, θ, z).

Here all the shear stresses and strains are zero, and the only non-zero displacement is u_r .

To solve this problem, we use the potential function approach. For this problem, consider

$$\varphi(r) = C_1 \ln \frac{r}{K} + C_2 r^2$$

$$u_i = \frac{1}{2\mu}\varphi_{,i}$$

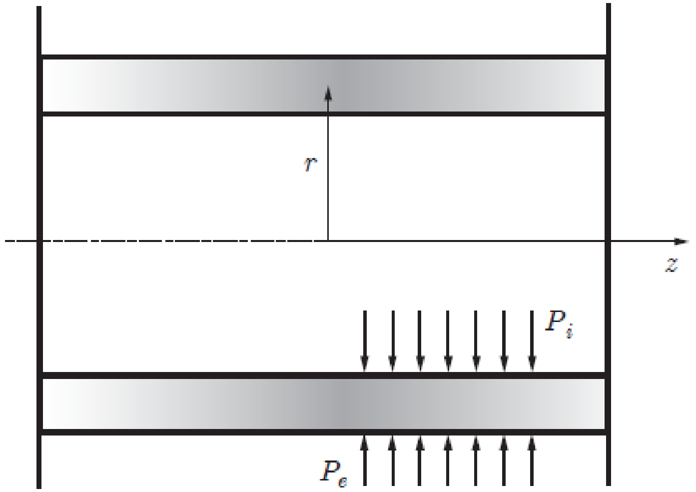
$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2\mu}\varphi_{,ij}$$

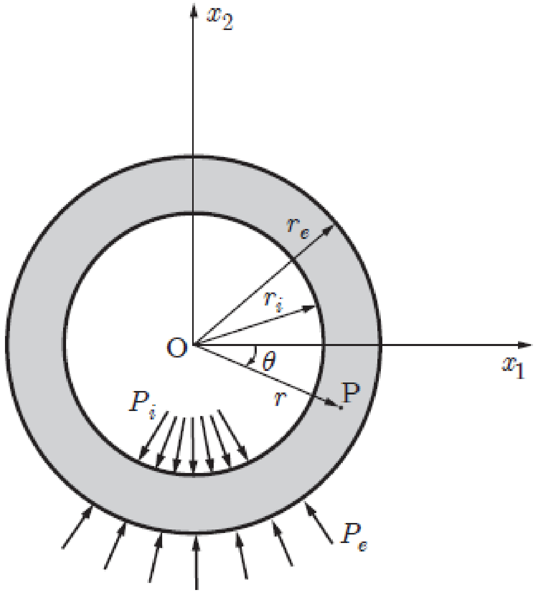
$$\varepsilon_{kk} = u_{k,k} = \frac{1}{2\mu}\varphi_{,kk}$$

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} = \frac{\lambda}{2\mu}\varphi_{,kk}\delta_{ij} + \varphi_{,ij} \ .$$

Potential or Displacement Functions

Example: *Hollow Cylinder with Internal and External Pressures and fixed Ends Subject to Internal and External Pressure* (THICK-WALLED CYLINDER)





$$\sigma_{rr} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e + \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

$$\sigma_{\theta\theta} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e - \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

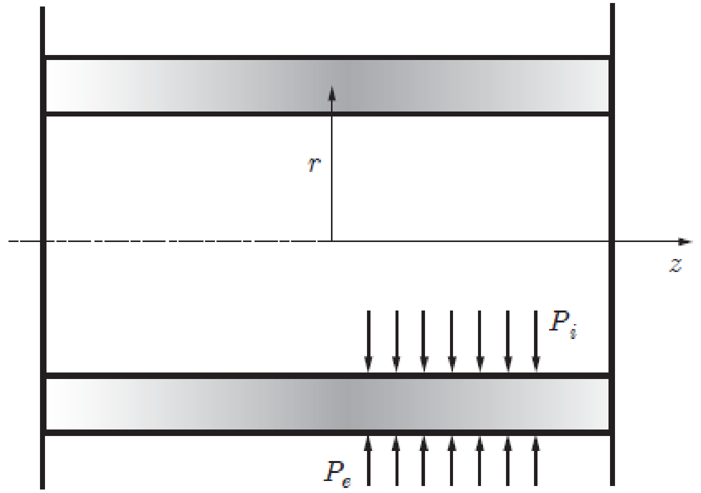
$$\sigma_{zz} = 2\nu \frac{r_i^2 P_i - r_e^2 P_e}{r_e^2 - r_i^2}$$

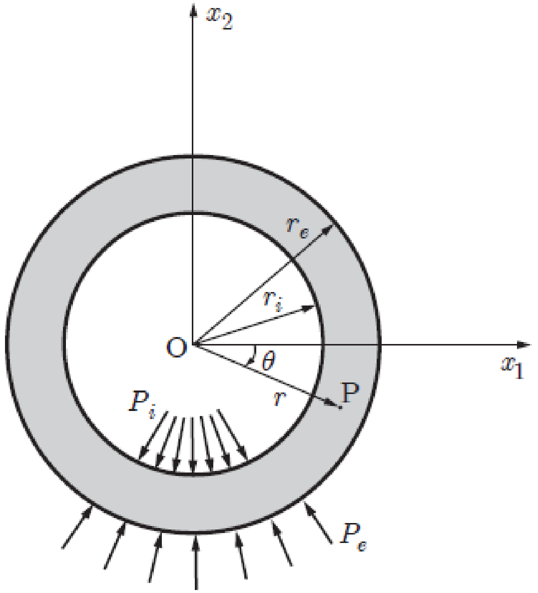
$$\sigma_{r\theta} = \sigma_{rz} = \sigma_{\theta z} = 0$$

$$u_r = \frac{r}{2\mu}\left(-\frac{1}{r^2}\frac{r_e^2 r_i^2 (P_e - P_i)}{r_e^2 - r_i^2} + \frac{r_i^2 P_i - r_e^2 P_e}{r_e^2 - r_i^2}(1 - 2\nu)\right)$$

APPROXIMATION FOR THIN-WALLED CONTAINER

Example: *Hollow Cylinder with Internal and External Pressures*
(THICK-WALLED CYLINDER to THIN-WALLED CYLIDER)





If the wall thickness is les than 10% of the Inner radius, the cylinder is classified as a thin-walled.

The variation of stress with radius is disregarded and the following approximation can be adopted:

$$e = r_e - r_i \quad e \ll r_i$$

$$r_e^2 - r_i^2 = (r_e - r_i)(r_e + r_i) \approx 2er_i$$
$$r_i^2 P_i - r_e^2 P_e \approx r_i^2 (P_i - P_e)$$
$$r_e^2 \approx r_i^2 \qquad r^2 \approx r_i^2 .$$

$$\sigma_{rr} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e + \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$
$$\sigma_{\theta\theta} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e - \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

$$\sigma_{rr} \approx 0$$
$$\sigma_{\theta\theta} \approx \frac{r_i(P_i - P_e)}{e} .$$

Galerkin Vector

Define a vector $\boldsymbol{V}$ function related to the displacement vector $\boldsymbol{u}$ with:

$$2\mu\boldsymbol{u} = 2(1-\nu)\nabla^2\boldsymbol{V} - \boldsymbol{\nabla}(\operatorname{div}\boldsymbol{V})$$

Introducing this in the Navier Eqs:

$$(\lambda+\mu)\nabla(\operatorname{div}\boldsymbol{u}) + \mu\Delta\boldsymbol{u} + \boldsymbol{f} = 0$$

and using the identities:

$$\operatorname{div}(\boldsymbol{\nabla}\Phi) = \nabla^2\Phi$$

$$\operatorname{div}(\boldsymbol{\nabla}^2\boldsymbol{a}) = \nabla^2(\operatorname{div}\boldsymbol{a})$$

$$\nabla^2(\boldsymbol{\nabla}\Phi) = \boldsymbol{\nabla}(\nabla^2\Phi)$$

$$2(1-\nu) = (\lambda+2\mu)/(\lambda+\mu)$$

$$\nabla^2\left(\nabla^2\boldsymbol{V}\right) = 0$$

Any biharmonic vector function can serve as a Galerkin vector.

The displacement $\boldsymbol{u}$ will satisfy Navier Eqs.

The identity and Navier Eqs are equivalent.

LOVE'S STRAIN FUNCTION

Particular case of the Galerkin vector

$$\boldsymbol{V} = V_3 \boldsymbol{e}_3$$

Introduce it in

$$\nabla^2 \left(\nabla^2 \boldsymbol{V}\right) = 0$$

$$\longrightarrow \nabla^2 \left(\nabla^2 V_3\right) = 0$$

Introduce it in

$$2\mu \boldsymbol{u} = 2(1-\nu)\nabla^2 \boldsymbol{V} - \boldsymbol{\nabla}(\operatorname{div} \boldsymbol{V})$$

$$2\mu \boldsymbol{u} = 2(1-\nu)(\nabla^2 V_3)\boldsymbol{e}_3 - \boldsymbol{\nabla}\left(\frac{\partial V_3}{\partial x_3}\right)$$

Displacement components:

$$2\mu u_1 = -\frac{\partial^2 V_3}{\partial x_1 \partial x_3}, \quad 2\mu u_2 = -\frac{\partial^2 V_3}{\partial x_2 \partial x_3}$$

$$2\mu u_3 = 2(1-\nu)\nabla^2 V_3 - \frac{\partial^2 V_3}{\partial x_3^2}$$

In cylindrical coordinates:

$$2\mu u_r = -\frac{\partial^2 V_z}{\partial r \partial z}, \quad 2\mu u_\theta = -\frac{1}{r}\frac{\partial^2 V_z}{\partial \theta \partial z}$$

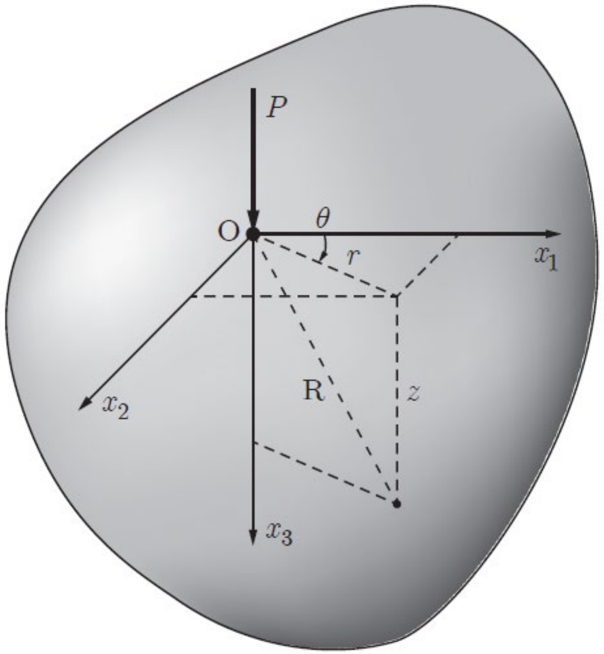
$$2\mu u_3 = 2(1-\nu)\nabla^2 V_3 - \frac{\partial^2 V_3}{\partial x_3^2}$$

KELVIN'S PROBLEM:

Concentrated Force Inside an Infinite Body

(application of the strain potential)

Infinite solid subjected to a concentrated force (known as Kelvin's Problem)



The solution to this problem is done by considering the Love's strain potential (Independent of angle θ, due to angular symmetry).

$$V_z = V_z(r, z)$$

Using the strain-displacement and stress-strain relations, the stresses are:

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2 V_z - \frac{\partial^2 V_z}{\partial r^2}\right)$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2 V_z - \frac{1}{r}\frac{\partial V_z}{\partial r} - \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2}\right)$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 V_z - \frac{\partial^2 V_z}{\partial r^2}\right)$$

$$\sigma_{r\theta} = -\frac{\partial^3}{\partial r\partial\theta\partial z}\left(\frac{V_z}{r}\right)$$

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial}{\partial \theta}\left((1-\nu)\nabla^2 V_z - \frac{\partial^2 V_z}{\partial z^2}\right)$$

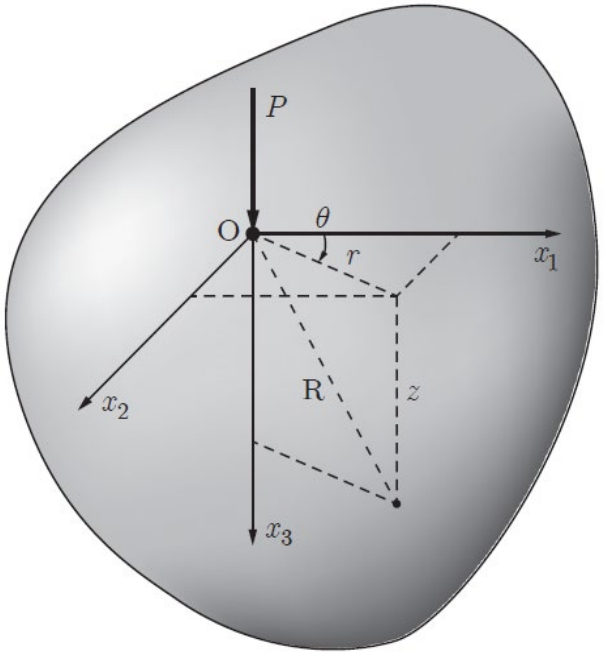
$$\sigma_{zr} = \frac{\partial}{\partial r}\left((1-\nu)\nabla^2 V_z - \frac{\partial^2 V_z}{\partial z^2}\right) .$$

KELVIN'S PROBLEM:

Concentrated Force Inside an Infinite Body

(application of the strain potential)

Infinite solid subjected to a concentrated force (Kelvin's Problem)



The following function

$$V_z = K(r^2 + z^2)^{1/2}$$

is found to satisfy

$$\nabla^2\left(\nabla^2 V_3\right) = 0$$

and its derivatives entering the stress components:

$$2\mu u_r = \frac{Krz}{(r^2+z^2)^{3/2}}, \quad 2\mu u_\theta = 0,$$

$$2\mu u_z = K\left[\frac{2(1-2\nu)}{(r^2+z^2)^{1/2}} + \frac{1}{(r^2+z^2)^{1/2}} + \frac{z^2}{(r^2+z^2)^{3/2}}\right]$$

KELVIN'S PROBLEM:

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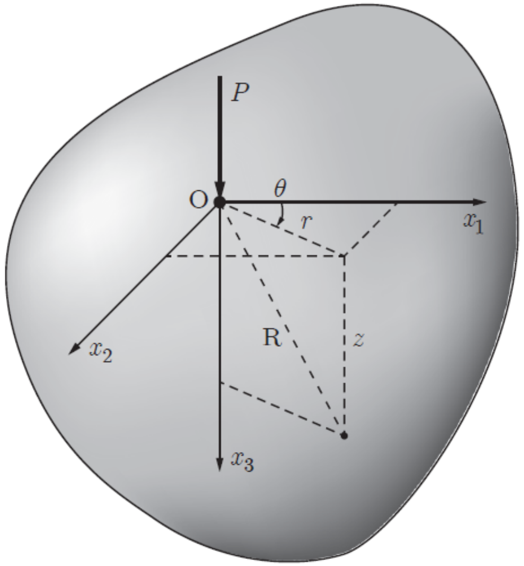
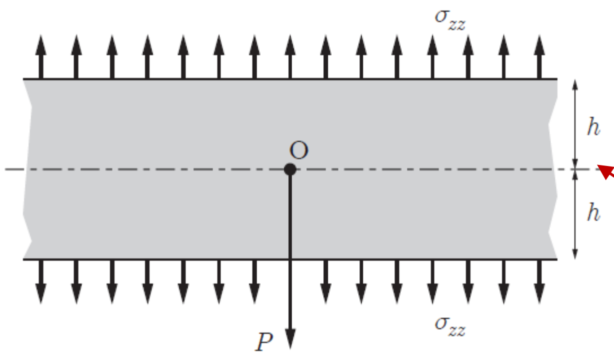




Stress components:

$$\sigma_{rr} = K\left[\frac{(1-2\nu)z}{(r^2+z^2)^{3/2}} - \frac{3r^2 z}{(r^2+z^2)^{5/2}}\right]$$

$$\sigma_{\theta\theta} = \frac{(1-2\nu)Kz}{(r^2+z^2)^{3/2}}$$

$$\sigma_{zz} = -K\left[\frac{(1-2\nu)z}{(r^2+z^2)^{3/2}} + \frac{3z^3}{(r^2+z^2)^{5/2}}\right]$$

$$\sigma_{rz} = -K\left[\frac{(1-2\nu)r}{(r^2+z^2)^{3/2}} + \frac{3rz^2}{(r^2+z^2)^{5/2}}\right]$$

$$\sigma_{r\theta} = \sigma_{\theta z} = 0\ .$$

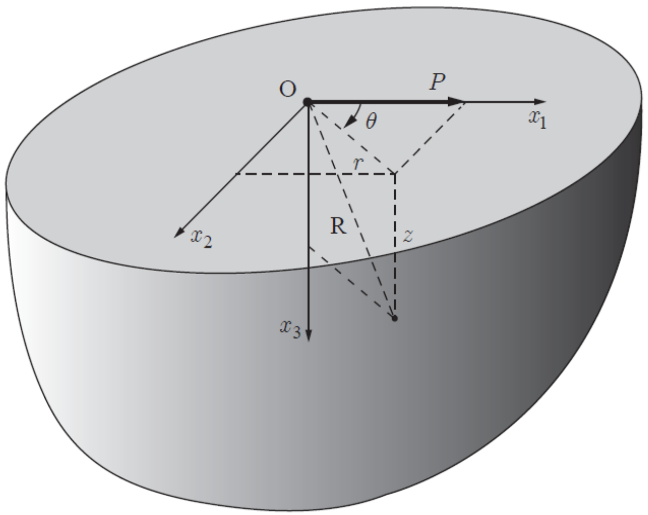
The constant K is calculated by considering the force equilibrium in the vertical axis:

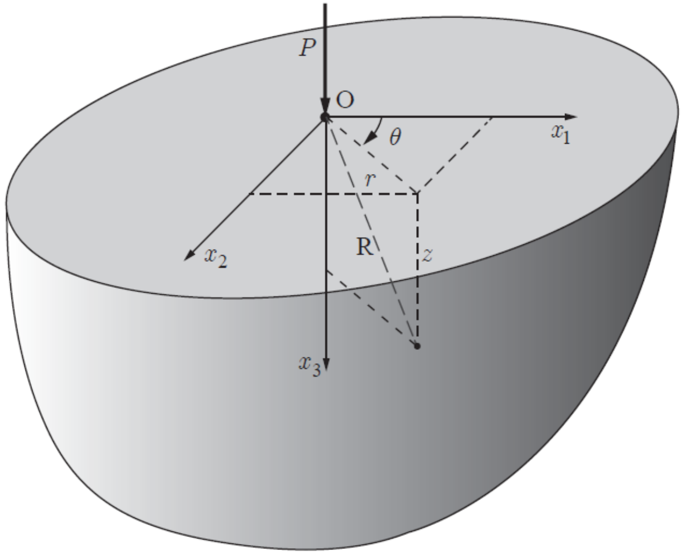
$$K = \frac{P}{8\pi(1-\nu)} \leftarrow P = \int_0^\infty 2\pi\, r dr \sigma_{zz}|_{z=-h} - \int_0^\infty 2\pi\, r dr \sigma_{zz}|_{z=+h}$$

CERRUTI'S PROBLEM:
Tangential Force at the end of a Semi-Infinite Body

BOUSSINSEQ'S PROBLEM:
Vertical Force at the End of a Semi-Infinite Body

The two important 3D problems are solved using potentials.





Problems of practical importance involve the effect of stresses acting in semi-infinite media (contact). Theoretically such solutions are obtained with the help of these two problems.

Potential or Stress Functions

To solve the Beltrami-Michel compatibility equations for the stresses we introduce a symmetric tensor stress function $\boldsymbol{\Phi}(\boldsymbol{x})$ (it is a tensor because the stress is a tensor) and expresses the stress components as follows:

$$\sigma_{11} = \frac{\partial^2 \Phi_{22}}{\partial x_3^2} + \frac{\partial^2 \Phi_{33}}{\partial x_2^2} - 2\frac{\partial^2 \Phi_{23}}{\partial x_2 \partial x_3}$$

$$\sigma_{22} = \frac{\partial^2 \Phi_{33}}{\partial x_1^2} + \frac{\partial^2 \Phi_{11}}{\partial x_3^2} - 2\frac{\partial^2 \Phi_{31}}{\partial x_3 \partial x_1}$$

$$\sigma_{33} = \frac{\partial^2 \Phi_{11}}{\partial x_2^2} + \frac{\partial^2 \Phi_{22}}{\partial x_1^2} - 2\frac{\partial^2 \Phi_{12}}{\partial x_1 \partial x_2}$$

$$\sigma_{12} = \frac{\partial^2 \Phi_{23}}{\partial x_3 \partial x_1} + \frac{\partial^2 \Phi_{31}}{\partial x_3 \partial x_2} - \frac{\partial^2 \Phi_{33}}{\partial x_1 \partial x_2} - \frac{\partial^2 \Phi_{12}}{\partial x_3^2}$$

$$\sigma_{23} = \frac{\partial^2 \Phi_{31}}{\partial x_1 \partial x_2} + \frac{\partial^2 \Phi_{12}}{\partial x_1 \partial x_3} - \frac{\partial^2 \Phi_{11}}{\partial x_2 \partial x_3} - \frac{\partial^2 \Phi_{23}}{\partial x_1^2}$$

$$\sigma_{31} = \frac{\partial^2 \Phi_{12}}{\partial x_2 \partial x_3} + \frac{\partial^2 \Phi_{23}}{\partial x_2 \partial x_1} - \frac{\partial^2 \Phi_{22}}{\partial x_3 \partial x_1} - \frac{\partial^2 \Phi_{31}}{\partial x_2^2}$$

With zero body forces these components satisfy equilibrium.

There are two types of functions:

If we consider only the diagonal elements Φ_{ii} we have the Maxwell system. If we keep the off diagonal elements we have the Morera system. Both systems satisfy equilibrium.

POTENTIAL OR STRESS FUNCTIONS

Plane stress problems:
we consider $\Phi_{33} = \Phi_{33}(x_1, x_2)$ which we call the Airy stress function. Some of the equations are not satisfied due to the approximate nature of Plane Stress but if we define the stresses as derivatives of:

$$\sigma_{11} = \frac{\partial^2 \Phi_{33}}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 \Phi_{33}}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2 \Phi_{33}}{\partial x_1 \partial x_2}$$

$$\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$$

The following biharmonic equation holds

$$\Delta\Delta\Phi_{33} = \frac{\partial^4 \Phi_{33}}{\partial x_1^4} + 2\frac{\partial^4 \Phi_{33}}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi_{33}}{\partial x_2^4} = 0$$

Plane strain problems:
Because $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$ in such problems, we introduce:

$$\Phi_{11} = \Phi_{22} = \nu\Phi_{33} \quad \text{and} \quad \Phi_{12} = \Phi_{23} = \Phi_{31} = 0$$

and the above biharmonic equation still holds true.

Beltrami-Michell compatibility equations

$$\nabla^2 \sigma_{11} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_{kk}}{\partial x_1^2} = 0$$

$$\nabla^2 \sigma_{22} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_{kk}}{\partial x_2^2} = 0$$

$$\nabla^2 \sigma_{33} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_{kk}}{\partial x_3^2} = 0$$

$$\nabla^2 \sigma_{12} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_{kk}}{\partial x_1 \partial x_2} = 0$$

$$\nabla^2 \sigma_{23} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_{kk}}{\partial x_2 \partial x_3} = 0$$

$$\nabla^2 \sigma_{31} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_{kk}}{\partial x_3 \partial x_1} = 0$$

FORMS OF AIRY STRESS FUNCTIONS

It is relatively easy to find a stress function that satisfies

$$\Delta\Delta\Phi_{33} = \frac{\partial^4\Phi_{33}}{\partial x_1^4} + 2\frac{\partial^4\Phi_{33}}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\Phi_{33}}{\partial x_2^4} = 0$$

But it is difficult to satisfy the boundary conditions even for functions that satisfy the biharmonic equation.

In general we start with (replaced $\Phi_{33}(x_1, x_2)$ with $\Phi(x_1, x_2)$)

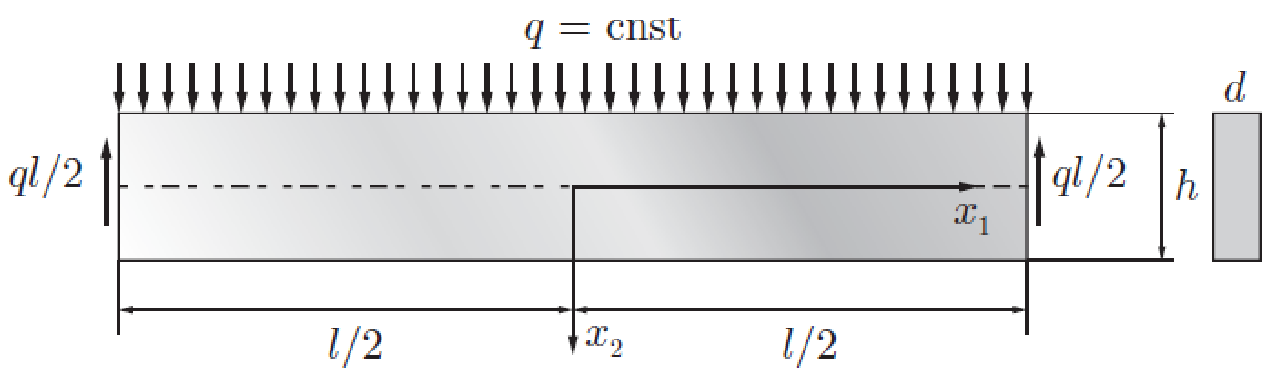
$$\Phi(x_1, x_2) = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + b_1 x_1^3 + b_2 x_1^2 x_2 + b_3 x_1 x_2^2 + b_4 x_2^3$$
$$+ c_1 x_1^4 + c_2 x_1^3 x_2 + c_3 x_1^2 x_2^2 + c_4 x_1 x_2^3 + c_5 x_2^4 + \cdots$$

All polynomial terms of degree less than or equal to three satisfy the biharmonic equation. Terms of higher order should not be considered, but if they must be included, their coefficients should be chosen with care to satisfy the biharmonic equation. This approach is effective in many problems with rectangular domains..

Mechanics of Solids: Theory of Elasticity - Airy stress functions

Example: *Long, Thin Beam with a Uniform Load*

$$\Phi(x_1, x_2) = A x_2^3 \left(x_1^2 - \frac{x_2^2}{5} \right) +$$
$$B x_1^2 x_2 + C x_2^3 + D x_1^2$$



The boundary conditions are :

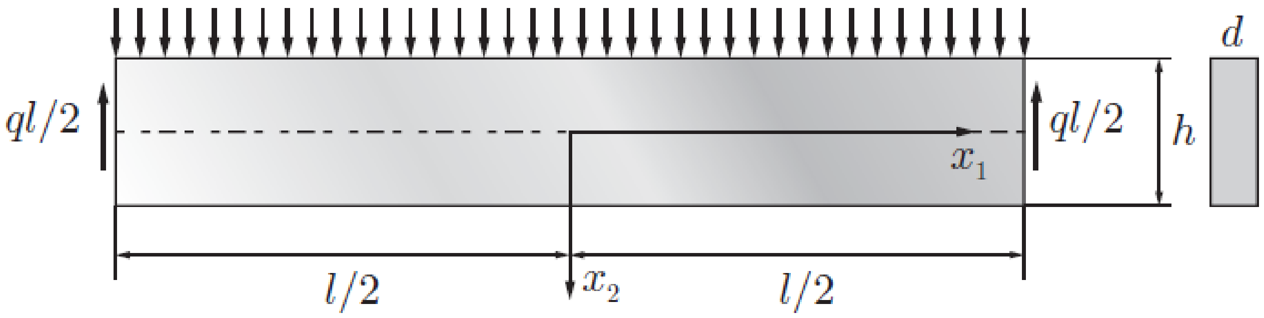
$$x_2 = -\frac{h}{2} \qquad \sigma_{22} = -q \qquad \sigma_{12} = 0$$
$$x_2 = \frac{h}{2} \qquad \sigma_{22} = 0 \qquad \sigma_{12} = 0\,.$$

At the two ends of the beam $x_1 = \pm l/2$:

$$N_1 = \int_{-h/2}^{h/2} \sigma_{11}\, dx_2 = 0$$
$$M_3 = \int_{-h/2}^{h/2} \sigma_{11} x_2\, dx_2 = 0$$
$$N_2 = \int_{-h/2}^{h/2} \sigma_{21}\, dx_2 = -\frac{ql}{2}$$

Mechanics of Solids: Theory of Elasticity - Airy stress functions

Example: *Long, Thin Beam with a Uniform Load*



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$$N_1 = \int_{-h/2}^{h/2} \sigma_{11}\, dx_2 = 0 \qquad N_2 = \int_{-h/2}^{h/2} \sigma_{21}\, dx_2 = -\frac{ql}{2}$$

$$M_3 = \int_{-h/2}^{h/2} \sigma_{11} x_2\, dx_2 = 0$$

We consider the Stress Function

$$\Phi(x_1, x_2) = A x_2^3 \left(x_1^2 - \frac{x_2^2}{5} \right) + B x_1^2 x_2 + C x_2^3 + D x_1^2$$

And use

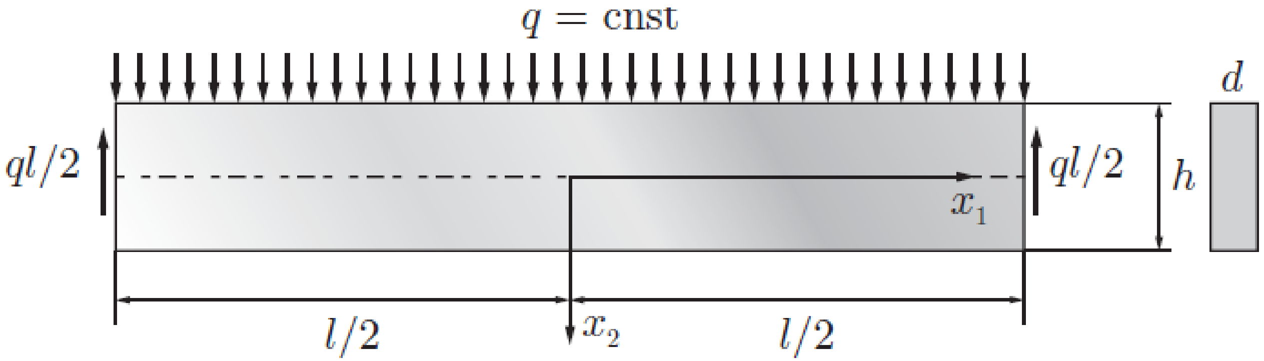
$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial {x_2}^2}\,, \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial {x_1}^2}\,, \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}$$

$$\begin{aligned} \sigma_{11} &= 6A x_2 x_1^2 - 4A x_2^3 + 6C x_2 \\ \sigma_{22} &= 2A x_2^3 + 2B x_2 + 2D \\ \sigma_{12} &= -6A x_2^2 x_1 - 2B x_1\,. \end{aligned}$$

$$A = -\frac{q}{h^3} \qquad B = \frac{3q}{4h} \qquad D = -\frac{q}{4}$$

$$C = \frac{q}{24 I_3} \left(\frac{l^2}{2} - \frac{h^2}{5} \right)$$

Example: *Long, Thin Beam with a Uniform Load*



$$\sigma_{11} = \frac{q}{2I_3} x_2 \left(\frac{l^2}{4} - \frac{x_1^2}{2}\right) + \frac{q}{2I_3} x_2 \left(\frac{2}{3}x_2^2 - \frac{h^2}{20}\right)$$

$$\sigma_{22} = -\frac{q}{2I_3}\left(\frac{x_2^3}{3} - \frac{h^2 x_2}{4} + \frac{h^3}{12}\right)$$

$$\sigma_{12} = -\frac{q}{2I_3} x_1 \left(\frac{h^2}{4} - x_2^2\right).$$

$$f = \frac{5}{384}\frac{ql^4}{EI_3}\left(1 + \frac{12}{5}\frac{h^2}{l^2}\left(\frac{4}{5} + \frac{\nu}{2}\right)\right)$$

With the stresses known, we use the strain stress relations to obtain the strains (Plane Stress)

$$\text{At} \quad x_1, x_2 = 0 \quad \Rightarrow \quad u_1 = 0, \quad u_2 = f, \quad \frac{\partial u_2}{\partial x_1} = 0$$

(where f is the maximum deflection)

$$\text{At} \quad x_1 = \pm\frac{l}{2}, \quad x_2 = 0 \Rightarrow \quad u_2 = 0$$

We integrate the strains to get the displacements

$$u_1 = \frac{q}{2EI_3}\left(\left(\frac{l^2 x_1}{4} - \frac{x_1^3}{3}\right)x_2 + \left(\frac{2x_2^3}{3} - \frac{h^2 x_2}{10}\right)x_1 + \nu\left(\frac{x_2^3}{3} - \frac{h^2 x_2}{4} + \frac{h^3}{12}\right)x_1\right)$$

$$u_2 = -\frac{q}{2EI_3}\left(\frac{x_2^4}{12} - \frac{h^2 x_2^2}{8} + \frac{h^3 x_2}{12} + \nu\left(\left(\frac{l_2}{4} - x_1^2\right)\frac{x_2^2}{2} + \frac{x_2^4}{6} - \frac{h^2 x_2^2}{20}\right)\right) - \frac{q}{2EI_3}\left(\frac{l^2 x_1^2}{8} - \frac{x_1^4}{12} - \frac{h^2 x_1^2}{20} + \left(1 + \frac{1}{2}\nu\right)\frac{h^2 x_1^2}{4}\right) + f.$$

FORMS OF AIRY STRESS FUNCTIONS IN CYLINDRICAL COORDINATES

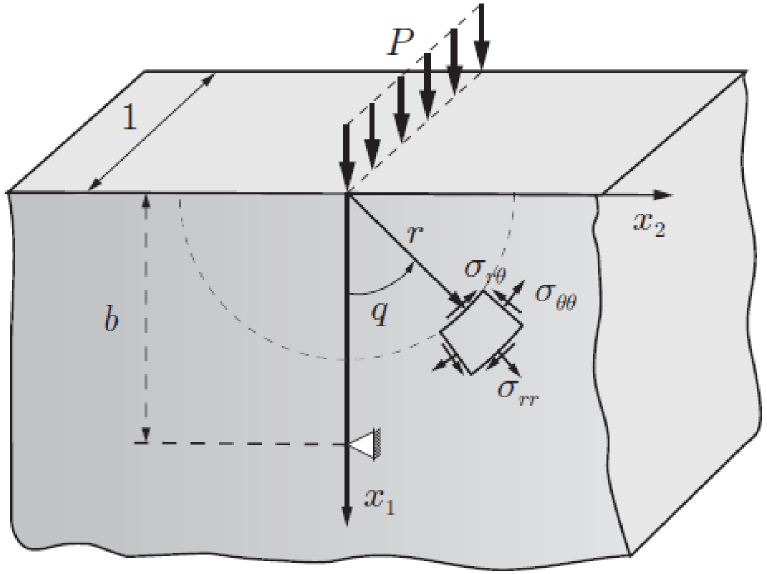
For problems with rotational symmetry we have
In general we start with (replaced $\Phi_{33}(r,\theta)$ with $\Phi(r,\theta)$)

$$
\begin{aligned}
\Phi(r,\theta) &= A_0 + A_1 \ln r + A_2 r^2 + A_3 r^2 \ln r \\
&+ (A_4 + A_5 \ln r + A_6 r^2 + A_7 r^2 \ln r)\theta \\
&+ \left(A_{11} r + A_{12} r \ln r + \frac{A_{13}}{r} + A_{14} r^3 + A_{15} r\theta + A_{16} r\theta \ln r \right) \cos\theta \\
&+ \left(B_{11} r + B_{12} r \ln r + \frac{B_{13}}{r} + B_{14} r^3 + B_{15} r\theta + B_{16} r\theta \ln r \right) \sin\theta \\
&+ \sum_{n=2}^{\infty} \left(A_{n1} r^n + A_{n2} r^{2+n} + A_{n3} r^{-n} + A_{n4} r^{2-n} \right) \cos n\theta \\
&+ \sum_{n=2}^{\infty} \left(B_{n1} r^n + B_{n2} r^{2+n} + B_{n3} r^{-n} + B_{n4} r^{2-n} \right) \sin n\theta
\end{aligned}
$$

The coefficients are constants and n is an integer.

Mechanics of Solids: Theory of Elasticity - Linear Load

Example: Normal Linear Load on the flat Edge of a Semi-Infinite Plate



Boundary Conditions

The stresses

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0 \quad \text{for} \quad \theta = \pm\pi/2$$

The vertical force is in equilibrium with the vertical component of the radial stress at a distance *r*.

The solution to the problem is obtained by setting the following Airy function

$$\Phi(r,\theta) = Cr\theta \sin\theta$$

Which satisfies the biharmonic equation

$$\nabla^4\Phi = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \cdot \left(\frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}\right) = 0$$

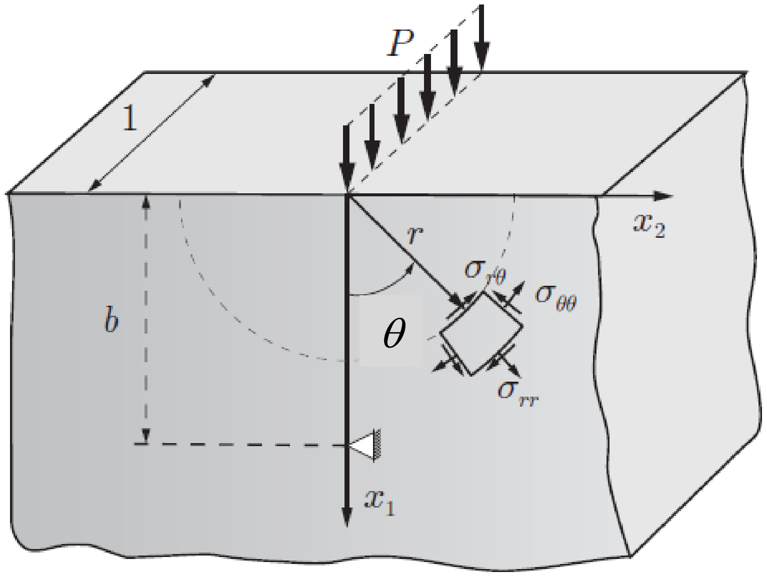
STRESS COMPONENTS

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} \qquad \sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta}$$

Mechanics of Solids: Theory of Elasticity - Linear Load

Example: Normal Linear Load on the flat Edge of a Semi-Infinite Plate



Boundary Conditions

The stresses are :

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0 \quad \text{for} \quad \theta = \pm\pi/2$$

The vertical force is in equilibrium with the vertical component of the radial stress at a distance *r*.

$$\sigma_{rr} = \frac{2C\cos\theta}{r}, \qquad \sigma_{\theta\theta} = \sigma_{r\theta} = 0$$

$$P + \int_{-\pi/2}^{+\pi/2} \sigma_{rr} \cos\theta (r d\theta) =$$

$$P + 2C \int_{-\pi/2}^{+\pi/2} \cos^2\theta d\theta = 0 \Longrightarrow C = -\frac{P}{\pi}$$

$$\sigma_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r}$$

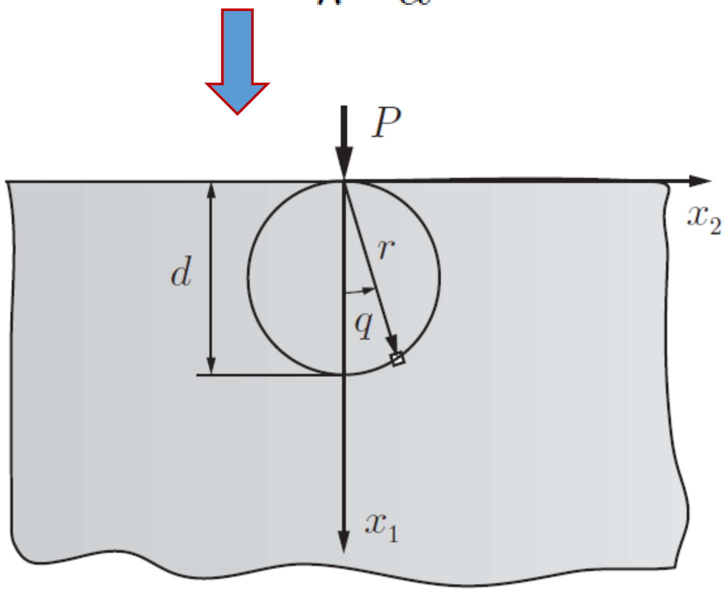
$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0$$

Mechanics of Solids: Theory of Elasticity - Linear Load

With $r = d\cos\theta$

$$\sigma_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r}$$

$$\sigma_{rr} = -\frac{2P}{\pi}\frac{1}{d}$$



From the stresses and Hook's law we have the strains,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = -\frac{2P}{\pi E}\frac{\cos\theta}{r}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{2P\nu}{\pi E}\frac{\cos\theta}{r}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = 0$$

They are integrated to get the displacements,

$$u_r = \frac{2P}{\pi E}\cos\theta \ln\frac{b}{r} - \frac{(1-\nu)P}{\pi E}\theta\sin\theta$$

$$u_\theta = \frac{(1+\nu)P}{\pi E}\sin\theta - \frac{2P}{\pi E}\sin\theta\ln\frac{b}{r} - \frac{(1-\nu)P}{\pi E}\theta\cos\theta$$

To avoid rigid body displacements, we impose the following conditions in the integration constant (b is an arbitrary constant):

$$u_\theta(r,\theta)|_{\theta=0} = 0 \qquad u_r(r,\theta)|_{\theta=0,r=b} = 0$$